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On the stability of plane shocks

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The stability of perturbed normal shock waves is considered. Shock perturbations depend directly upon the disturbances in the flow adjacent to the shock. In the present paper an initially stationary shock is assumed to be perturbed by acoustic waves reaching it from the downstream side. This case corresponds to the situation occurring in shock diffraction or reflexion. Two-dimensional problems of this type have been investigated previously, both analytically and experimentally. These previous analytic results have, in all cases, indicated that the perturbations of the shock decay with time as $t^{-\frac{3}{2}}$, while experimentally both $t^{-\frac{1}{2}}$ and $t^{-\frac{3}{2}}$ decays have been observed. It is demonstrated in the present investigation that, when waves are continuously generated at a point or points behind the shock, a $t^{-\frac{1}{2}}$ decay of the shock perturbations will occur, corresponding to the decay of the incident waves. However, when the source of waves is located only at the shock, as in a diffraction problem, $t^{-\frac{3}{2}}$ decay occurs owing to the cancellation, to lowest order, of the incident wave by its reflexion from the shock. These results explain the divergence between theory and experiment in this area, since the experiments giving the slower decay contained a source of waves behind the shock.

It is concluded that shock stability can only be considered in the context of the type of disturbances incident upon the shock.

1. Introduction

Although the stability of a shock is of fundamental importance in many applications of gasdynamics, this stability has not been fully investigated in even the simplest of cases. Most previous investigations have considered a specific case and determined a decay law for that particular case, but no general results are available for the decay of shock perturbations. Indeed, the phenomena leading to decay are not clearly understood.

A shock discontinuity adjusts instantaneously to the adjacent upstream and downstream flow conditions. Thus shock perturbations are dependent on the perturbations incident on the shock from the adjacent flow. Consider a shock in the absence of solid boundaries. One type of shock perturbation may be caused

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by entropy or vorticity disturbances which are convected by the flow. These can only reach a stationary shock from upstream. Pressure disturbances, however, propagate through the fluid and can reach the shock from either side. Any acoustic pressure disturbance upstream of the shock will reach the shock and interact with it. On the downstream side of the shock only that class of waves whose component of propagation velocity normal to the shock is greater than the flow velocity downstream of the shock will reach the shock.

These two basic types of disturbances may be divided into three classes: entropy and vorticity disturbances upstream of the shock, pressure waves upstream of the shock and pressure waves downstream of the shock. The latter of these three cases, corresponding to the perturbations of a shock moving into a uniform medium, will be considered below. The pressure waves will be assumed to be weak and may be either cylindrical or spherical. The interaction of plane waves with a normal shock has been considered previously by Moore (1954) and does not lead to decaying shock perturbations.

Previously work on shock stability, with one exception, has been concerned with shock stability in two-dimensional cases and, thus, with cylindrical waves behind the shock. Although not directly concerned with stability, Lighthill's (1949) investigation of the diffraction of a blast forms the basis for much of the subsequent work. Freeman (1955) made use of Lighthill's approach in his investigation of the shock produced when a corrugated piston is moved impulsively through a gas. He found that the corrugations in the shock decay with time as $t^{-\frac{3}{2}}$. Similar problems involving corrugated walls and pistons were considered by Zaidel (1960), Nikolaev (1965) and Kovitz & Briscoe (1968). All of these investigators obtained the $t^{-\frac{3}{2}}$ decay law. This decay law was later verified experimentally by Briscoe & Kovitz (1968). The second problem which has been considered is the passage of a normal shock over an obstacle on the wall of a shock tube. This problem was first considered by Freeman (1957), who, again, predicted $t^{-\frac{3}{2}}$ decay. Experiments, however, did not verify this result. Lapworth (1959) reasoned that separation of the unsteady boundary layer from the wall at the obstacle, an effect not considered in the theory, would adversely effect his experiment and, thus, used only data obtained when the shock was near the obstacle to minimize this effect. Yet only 'fair' agreement with Freeman's theoretical results could be obtained. Later Bowman (1966) reanalysed all of Lapworth's data and found a $t^{-\frac{1}{2}}$ decay.

The previous theoretical analyses assumed that the disturbances were generated at the shock and subsequently propagated along the shock through the fluid behind it. This led to a $t^{-\frac{3}{2}}$ decay for cylindrical waves. The experimental measurements were less definite with both $t^{-\frac{1}{2}}$ and $t^{-\frac{3}{2}}$ decays having been observed. The one-half-power decay occurred when additional disturbances were generated behind the shock.

In the present analysis the disturbances are assumed to originate at an arbitrary moving point. In this case the three-halves-power decay is obtained when the origin of the waves is at the shock. However, if the waves originate behind the shock a one-half-power initial decay occurs when the waves first reach the shock, later becoming a three-halves-power decay as the waves interact with the shock at greater angles. Thus, if waves are continuously reaching the shock, the slower initial decay will dominate and lead to a one-half-power decay of the shock perturbations in the two-dimensional case.

The change in the decay mechanism will be shown to result from the increasing angle of incidence between the shock and an intersecting cylindrical (or spherical) wave generated behind it. The local reflexion coefficient for reflexion of a pressure wave from a shock varies strongly with the angle of incidence. At small angles of incidence the reflected wave is very weak and the incident wave decays in a geometric manner, like $t^{-\frac{1}{2}}$ for cylindrical waves or t^{-1} for spherical waves. As the angle of incidence of the wave approaches its maximum the reflected wave becomes strong, which results in a cancellation with the leading term in a progressing-wave expansion for the incident wave and yields $t^{-\frac{3}{2}}$ decay for cylindrical waves or t^{-2} decay for spherical waves.

2. Basic equations and solutions

In the flow behind the shock the propagation of weak waves is governed by the equations obtained by linearizing the equations of conservation of mass, momentum and energy:

$$\frac{\partial \rho}{\partial t}+U\frac{\partial \rho}{\partial x}+\rho_1\frac{\partial u}{\partial x}+\rho_1\frac{\partial v}{\partial y}+\rho_1\frac{\partial w}{\partial z}=0, \tag{1a}$$

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+\frac{1}{\rho_1}\frac{\partial P}{\partial x}=0, \tag{1b}$$

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+\frac{1}{\rho_1}\frac{\partial P}{\partial y}=0, \tag{1c}$$

$$\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}+\frac{1}{\rho_1}\frac{\partial P}{\partial z}=0, \tag{1d}$$

$$\frac{\partial P}{\partial t}+U\frac{\partial P}{\partial x}-a_1^2\left(\frac{\partial \rho}{\partial t}+U\frac{\partial \rho}{\partial x}\right)=0, \tag{1e}$$

where the co-ordinates have been chosen so that the unperturbed shock is stationary (see figure 1), U, ρ_1 and a_1 are, respectively, the unperturbed flow velocity, density and sound speed behind the shock, and u, v, w, P and ρ are the velocity perturbations in the x, y and z directions, the pressure perturbation and the density perturbation. These equations may be combined into the single equation

$$\frac{1}{a_1^2}\frac{\partial^2 P}{\partial t^2}+2\frac{U}{a_1^2}\frac{\partial^2 P}{\partial t\,\partial x}-\beta^2\frac{\partial^2 P}{\partial x^2}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}=0, \tag{2}$$

where $\beta^2 = 1-U^2/a_1^2$. By making a Galilean transformation to co-ordinates (x_f, y_f, z_f) moving with the flow behind the unperturbed shock (see figure 2) in (2), the usual three-dimensional wave equation

$$\frac{1}{a_1^2}\frac{\partial^2 P}{\partial t^2}-\frac{\partial^2 P}{\partial x_f^2}-\frac{\partial^2 P}{\partial y_f^2}-\frac{\partial^2 P}{\partial z_f^2}=0 \tag{3}$$

is obtained.

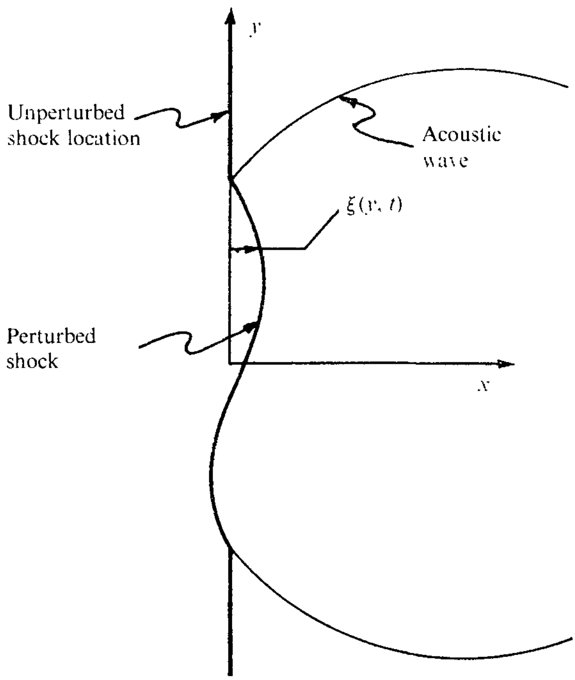

FIGURE 1. The shock perturbation.

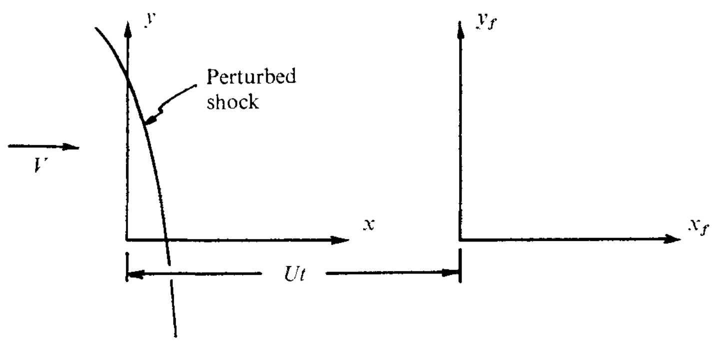

FIGURE 2. Flow-fixed and shock-fixed co-ordinates.

Two types of solutions of (3) will be considered.

(i) *Spherical progressing waves.*

$$P = (K_1 \sin l\phi + K_2 \cos l\phi)\, P_m^l(\cos\theta) \sum_{n=0}^{m} \frac{f_{m,n}(at-R)}{R^{1+n}}, \tag{4}$$

where $P_m^l(\cos\theta)$ is the associated Legendre function of the first kind with argument $\cos\theta, R^2 = x_f^2 + y_f^2 + z_f^2$, K_1 and K_2 are constants and

$$f_{n+1,m}(\zeta) = \frac{m(m+1)-n(n+1)}{2(n+1)} \int f_{n,m}(\zeta)\, d\zeta,$$

$f_{0,m}(\zeta)$ being an arbitrary function of ζ. Here R, θ and ϕ are spherical co-ordinates, ϕ being the azimuthal angle. The integers l and m are determined by the type of source present. The general form of progressive-wave solutions is discussed by Friedlander (1946). Using these ideas (4) is easily shown to be a solution of (3).

(ii) *Cylindrical progressing waves.*

$$P = (K_3 \sin m\theta + K_4 \cos m\theta) \sum_{n=0}^{\infty} \frac{f_{n,m}(at-r)}{r^{\frac{1}{2}+n}}, \tag{5}$$

where $r^2 = x_f^2 + y_f^2$, K_3 and K_4 are constants and

$$f_{n+1,m}(\zeta) = \frac{m^2-(n+\frac{1}{2})^2}{2(n+1)} \int f_{n,m}(\zeta)\, d\zeta,$$

$f_{0,m}(\zeta)$ being an arbitrary function of ζ. Here r and θ are plane polar co-ordinates. The integer m is determined by the type of source. This cylindrical solution is an asymptotic solution of the wave equation for $at-r \to 0$ and can be obtained from the integral solution of the two-dimensional wave equation (see Van Moorhem 1971). These two sets of solutions represent waves generated at a fixed source and propagating away from that source.

3. Shock boundary conditions

The shock forms a boundary for the disturbed region of flow. No waves can cross the shock as they would immediately be swept back into the shock. Following the approach used by Moore (1954), the perturbations to the flow variables behind the shock can be related to the shock perturbations by the expressions

$$P(0,y,t)/P_0 = A(M)\, a_0^{-1}\, \partial\xi/\partial t, \tag{6a}$$

$$u(0,y,t)/a_0 = B(M)\, a_0^{-1}\, \partial\xi/\partial t, \tag{6b}$$

$$v(0,y,t)/a_0 = C(M)\, \partial\xi/\partial y \tag{6c}$$

and

$$w(0,y,t)/a_0 = C(M)\, \partial\xi/\partial z, \tag{6d}$$

where

$$A(M) = -\frac{4\gamma}{\gamma+1} M, \quad B(M) = \frac{2}{\gamma+1}\left(\frac{M^2+1}{M^2}\right)$$

and

$$C(M) = \frac{2}{\gamma+1}\left(\frac{M^2-1}{M}\right).$$

P_0 and a_0 are the pressure and sound speed upstream of the shock, M is the shock Mach number V/a_0, and ξ is the shock perturbation (see figure 1).

It is possible to combine these expressions relating the flow and shock perturbations with the flow equations to obtain a single expression governing the pressure perturbation at the shock. This gives

$$\frac{1}{v_w^2}\frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} = -\left(\frac{\gamma+1}{2\gamma}\right)\frac{P_0}{P_1}\frac{A(M)}{a_0}\frac{\partial^2 P}{\partial x\, \partial t} \tag{7}$$

on $x = 0$, where

$$\frac{v_w^2}{a_1^2} = \frac{[(\gamma-1)M^2+2][2\gamma M^2-(\gamma-1)](M^2-1)}{[2(2\gamma-1)M^4+(\gamma+5)M^2-(\gamma-1)](\gamma+1)}.$$

The derivation of this boundary condition and its implications are discussed by Van Moorhem (1971).

4. Interaction of the waves and shock

The interaction of plane waves with a shock has been investigated previously by Moore (1954), Brillouin (1955), Johnson & Laporte (1958), D'iakov (1958), Kontorovich (1959) and McKenzie & Westphal (1968). These investigators found an angle-dependent reflexion coefficient, and a rather peculiar relationship between the angles of incident and reflected waves with a maximum angle of incidence occurring at less than 90°. This maximum results from the necessity that the component of the wave velocity normal to the shock be greater than the flow velocity.

The interactions of both cylindrical and spherical waves with a planar shock exhibit many of the characteristics of the plane-wave interaction, but are complicated by the possibility of motion of the source of the waves, and the difficulty of not knowing *a priori* the form of the reflected waves. Both of these difficulties can be overcome by application of the Lorentz transformation of co-ordinates as described by Sears (1954).

The Lorentz transformation

$$\left.\begin{aligned} x_f = (X+V_sT)/\beta_s, \quad y_f = Y, \quad z_f = Z,\\ t = (T+XV_s/a_1^2)/\beta_s, \end{aligned}\right\} \tag{8}$$

where $\beta_s^2 = 1 - V_s^2/a_1^2$, has the property that it leaves the wave equation (3) unchanged, while the moving point $(x_f, y_f, z_f) = (V_s t, 0, 0)$ is fixed in the Lorentz co-ordinates at $(X, Y, Z) = (0, 0, 0)$. If V_s is chosen as the velocity of the wave source with respect to the fluid behind the shock, the solution of the ordinary wave equation as given by (4) or (5) may be directly applied in the Lorentz co-ordinates. On transforming back to the physical co-ordinates the waves due to a moving three-dimensional source are obtained as

$$P = (K_1 \sin l\phi + K_2 \cos l\phi)\, P_m^l(\cos\Theta_s) \sum_{n=0}^{m} \frac{f_{n,m}(\Phi_s)}{s_s^{1+n}}, \tag{9}$$

where $$s_s^2 = (x_f - V_s t)^2 + \beta_s^2(y_f^2 + z_f^2), \quad \cos\Theta_s = (x_f - V_s t)/s_s$$

and $$\Phi_s = a_1 t - (V_s/a_1)\, x_f - s_s.$$

For a two-dimensional source

$$P = (K_3 \sin m\Theta_c + K_4 \cos m\Theta_c) \sum_{n=0}^{\infty} \frac{f_{n,m}(\Phi_c)}{s_c^{\frac{1}{2}+n}}, \tag{10}$$

where $$s_c^2 = (x_f - V_s t)^2 + \beta_s^2 y_f^2, \quad \cos\Theta_c = (x_f - V_s t)/s_c,$$

$$\sin\Theta_c = \beta_s y_f/s_c, \quad \Phi_c = a_1 t - (V_s/a_1)\, x_f - s_c.$$

The quantities s (either s_s or s_c) appear in the same manner as distance from the source for a stationary wave source, and Θ_s and Θ_c are angle-like variables which identify the direction of propagation of points on a wave front, but are not physical angles.

A second Lorentz transformation can now be made by choosing in this case $V_s = -U$, the velocity of the shock with respect to the flow behind the shock, so that the unperturbed shock is now fixed on the line $X = 0$ and the governing equation remains the ordinary wave equation. The shock boundary condition (7), after transformation, becomes

$$\frac{\hat{A}}{a_1^2}\frac{\partial^2 P}{\partial T^2} - \frac{\partial^2 P}{\partial Y^2} - \frac{\partial^2 P}{\partial Z^2} = \frac{\hat{B}}{a_1}\frac{\partial^2 P}{\partial X \partial T}, \tag{11}$$

where $\quad \hat{A} = (M^2+1)/M^2, \quad \hat{B} = 2\{[M^2(\gamma-1)+2]/[2\gamma M^2-(\gamma-1)]\}.$

The general form of the reflected wave is now easily obtained by noting that the wave equation is symmetric in X, that is, if $F(X, Y, Z)$ is a solution of the wave equation, then $F(-X, Y, Z)$ is also a solution. This procedure is a generalization of the method of images and determines the lines of constant phase of the reflected wave, ensuring that they connect with the lines of constant phase of the incident wave at the mean shock location. The amplitude of the reflected wave, however, remains to be determined from the shock boundary condition. Substitution of the expressions for the reflected and incident waves into the shock boundary condition leads to the conclusion that a single term of the general form of (4) or (5) is not sufficient to satisfy the boundary condition and the reflected wave must consist of a sum of terms of the form of (4) or (5) with the coefficients of the lowest-order terms in s^{-1} forming a Fourier series in the case of cylindrical waves or a generalized Fourier series in the case of spherical waves. Recasting the series representing the reflected wave, substituting both it and the incident wave into the boundary condition (11) and equating terms of equal order in the 'distance' s from the source gives, at the unperturbed shock location,

$$P = (K_1 \sin l\phi + K_2 \cos l\phi)\, P_m^l(\cos\Theta_{s,s})\,(1+\mathscr{R}(\Theta_{s,s}))f_{0,m}(\Phi_{s,s})/s_{s,s} + O(s_{s,s}^{-2}), \tag{12}$$

where

$$\Theta_{s,s} = \tan^{-1}\{\beta_s(y_f^2+z_f^2)^{\frac{1}{2}}/(U+V_s)\,t\}, \quad s_{s,s}^2 = (U+V_s)^2t^2+\beta_s^2(y^2+z^2)$$

and $$\Phi_{s,s} = (a_1+V_s U/a_1)\,t - s_{s,s},$$

for spherical waves and

$$P = (K_3 \sin m\Theta_{c,s} + K_4 \cos m\Theta_{c,s})\,(1+\mathscr{R}(\Theta_{c,s}))f_{0,m}(\Phi_{c,s})/s_{c,s}^{\frac{1}{2}} + O(s_{c,s}^{-\frac{3}{2}}), \tag{13}$$

where

$$\Theta_{c,s} = \tan^{-1}\{\beta_s y_f/(U+V_s)\,t\}, \quad s_{c,s}^2 = (U+V_s)^2t^2+\beta_s^2 y^2$$

and $$\Phi_{c,s} = (a_1+V_s U/a_1)\,t - s_{c,s},$$

for cylindrical waves. The quantity

$$\begin{aligned}\mathscr{R}(\Theta) \equiv\; & -1-\{2\hat{B}[a_1^2+V_sU-a_1(V_s+U)\cos\Theta]\,[a_1(V_s+U)-(a_1^2+V_sU)\cos\Theta]\} \\ & \times\{\hat{A}[a_1^2+V_sU-a_1(V_s+U)\cos\Theta]^2-a_1^2\beta^2\beta_s^2\sin\Theta \\ & -\hat{B}[a_1^2+V_sU-a_1(V_s+U)\cos\Theta]\,[a_1(V_s+U)-(a_1^2+V_sU)\cos\Theta]\}^{-1}\end{aligned} \tag{14}$$

is the reflexion coefficient found by Moore (1954) and others for the interaction of plane waves and shocks, but is expressed here in terms of Θ (either $\Theta_{s,s}$ or $\Theta_{c,s}$) rather than the physical angle θ_i. The reflexion coefficient is plotted in

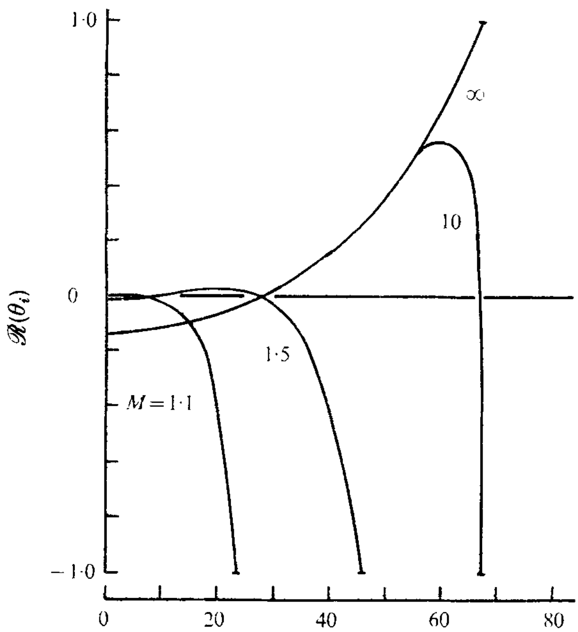

FIGURE 3. The reflexion coefficient.

figure 3 *vs*. the physical angle of incidence. For small angles of incidence the reflexion coefficient is seen to be small and nearly constant for all finite Mach numbers. The reflexion coefficient becomes -1 at the critical angle. This critical angle is the maximum angle at which an acoustic wave can reach the shock, $\theta_i = \cos^{-1}(U/a_1)$, and corresponds to $\Theta = \cos^{-1}[a_1(U+V_s)/(a_1^2+UV_s)]$.

Examination of the incident and reflected waves indicates that the angles of incidence and reflexion are related in the same manner as in the plane-wave case. The amplitude of the lowest-order term in s^{-1} of the reflected wave is determined by the reflexion coefficient for plane waves. The higher-order terms, however, do not follow this simple rule (Van Moorhem (1971) has determined the amplitude of the higher-order terms for cylindrical waves).

The shock perturbations may now be determined from (12) or (13) and (6*a*). The integration cannot be carried out exactly, however the asymptotic form is easily determined as

$$\xi = (a_0/P_0A)\{(K_1 \sin l\phi + K_2 \cos l\phi)\, P_m^l(\cos\Theta_{s,s}) \\ \times (1+\mathscr{R}(\Theta_{s,s}))\, s_{s,s}^{-1} \int f_{0,m}(\Phi_{s,s})\, dt + O(s_{s,s}^{-2})\} \quad (15)$$

for spherical waves or

$$\xi = (a_0/P_0A)\{(K_3 \sin m\Theta_{c,s} + K_4 \cos m\Theta_{c,s}) \\ \times (1+\mathscr{R}(\Theta_{s,s}))\, s_{c,s}^{-\frac{1}{2}} \int f_{0,m}(\Phi_{c,s})\, pt + O(s_{c,s}^{-\frac{3}{2}})\} \quad (16)$$

for cylindrical waves. It is clear that the shock perturbations decay in the same manner as the pressure perturbations at the shock.

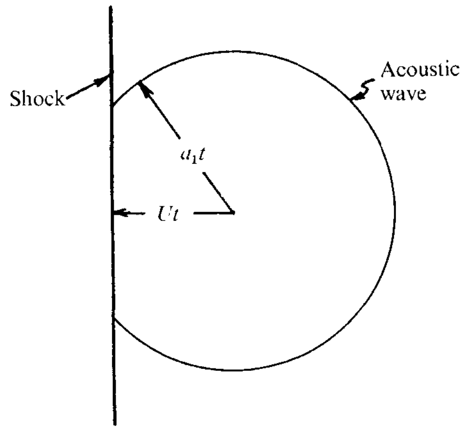

FIGURE 4. The intersection with the shock of a cylindrical or spherical wave emitted at the shock.

Both the decay of the pressure perturbations at the shock and the shock perturbations are determined by the angle of incidence of the wave fronts, or lines of constant phase, at the shock. To display the different modes of decay of these perturbations the behaviour of the cofficients of the lowest-order term in s^{-1} in each expansion must be determined on a line of constant Φ. The cylindrical-wave cases (13) and (16) will be considered in detail, the spherical cases (12) and (15) being very similar.

On a line of constant phase $\Phi_{c,s}$, all of the coefficients of $s_{c,s}^{-\frac{1}{2}}$ in (13) and (16) are dependent only on $\Theta_{c,s}$ and it is the behaviour of $\Theta_{c,s}$ which must first be determined. From (13), $\cos\Theta_{c,s}$ and $\sin\Theta_{c,s}$ can be expressed as

$$\cos\Theta_{c,s} = \frac{a_1(U+V_s)}{a_1^2+V_s U}\left(1+\frac{\Phi_{c,s}}{s_{c,s}}\right) \tag{17a}$$

and

$$\sin\Theta_{c,s} = \left[1-\frac{a_1^2(U+V_s)^2}{(a_1^2+V_s U)^2}\left(1+\frac{\Phi_{c,s}}{s_{c,s}}\right)^2\right]^{\frac{1}{2}}. \tag{17b}$$

Three cases are now of interest: a wave emitted at the shock, a wave emitted far behind the shock and just reaching it, and a wave emitted far behind the shock which has interacted with the shock over a long period of time.

The first case, a wave front emitted at the shock, can be shown from (10) to correspond to $\Phi_{c,s} = 0$, thus from (17)

$$\cos\Theta_{c,s} = a_1(U+V_s)/(a_1^2+V_s U) \tag{18a}$$

and

$$\sin\Theta_{c,s} = [1-a_1^2(U+V_s)^2/(a_1^2+V_s U)^2]^{\frac{1}{2}}. \tag{18b}$$

These give the value of $\Theta_{c,s}$ corresponding to the critical angle of incidence and to $\mathscr{R}(\Theta_{c,s}) = -1$. The same conclusion can be reached from figure 4, which shows a wave front emitted at the shock, and for which the angle of incidence is always equal to the critical angle $\theta_t = \cos^{-1}(U/a_1)$. The coefficient of $s_{c,s}^{-\frac{1}{2}}$ is, therefore, zero and the decay is determined by the next term in the expansion.

Since $\Theta_{c,s}$ is a constant in this case, the second term decays as $s_{c,s}^{-\frac{3}{2}}$ or $t^{-\frac{3}{2}}$ for large times.

The second case, a wave emitted behind the shock and just reaching it, corresponds to a non-zero value of $\Phi_{c,s}$. If the wave was emitted at time $t = t_0$ and location $x_f = 0$, $y_f = 0$, the value of $\Phi_{c,s}$ is found from (10) to be $(a_1 - V_s)\,t_0$. The value of $s_{c,s}$ when the wave initially reaches the shock can be defined to be s_{int} and is found from (13) as

$$s_{\text{int}} = [(U+V_s)\,\Phi_{c,s}]/[a_1(1+V_s U/a_1^2) + U + V_s].$$

The $\sin\Theta_{c,s}$ and $\cos\Theta_{c,s}$ can then be expanded in a series of powers of $(s_{c,s} - s_{\text{int}})/\Phi_{c,s}$, yielding

$$\cos\Theta_{c,s} = 1 + O\{(s_{c,s} - s_{\text{int}})/\Phi\} \tag{19a}$$

and
$$\sin\Theta_{c,s} = 2^{\frac{1}{2}} \frac{a_1(a_1^2 + V_s U) - a_1^2(U+V_s)}{[a_1^3(U+V_s)\,(a_1^2+V_s U)]^{\frac{1}{2}}} \left(\frac{s_{c,s} - s_{\text{int}}}{\Phi_{c,s}}\right)^{\frac{1}{2}} + \dots. \tag{19b}$$

Substituting (19) into (14) gives

$$\mathscr{R}(\Theta_{c,s}) = \frac{a_1\hat{A} - \hat{B}(a_1^2 + V_s U - (U+V_s)^2)}{a_1\hat{A} + \hat{B}(a_1^2 + V_s U - (U+V_s)^2)} \left[1 + O\left(\frac{s_{c,s} - s_{\text{int}}}{\Phi_{c,s}}\right)^{\frac{1}{2}}\right]. \tag{20}$$

Thus for $\Phi_{c,s} \gg s_{c,s} - s_{\text{int}}$ (waves emitted far behind the shock) and if K_4 is zero in (10), (13) and (16), so that no explicit $\sin\Theta_{c,s}$ terms occur, an initial $s_{c,s}^{-\frac{1}{2}}$ or $t^{-\frac{1}{2}}$ decay occurs (if K_4 is not zero a $[(s_{c,s} - s_{\text{int}})/s_{c,s}]^{\frac{1}{2}}$ behaviour occurs). This can be seen physically by considering a cylindrical wave emitted far behind the shock. When this wave reaches the shock it will initially be at a zero angle of incidence, the reflexion coefficient will be nearly constant and the wave will decay in a geometric manner, as $t^{-\frac{1}{2}}$.

The third case, that of a wave emitted behind the shock and interacting with it over a long period of time, again corresponds to non-zero values of $\Phi_{c,s}$ and to large values of $s_{c,s}$. Assuming $\Phi_{c,s} \ll s_{c,s}$, equation (17) can be expressed as

$$\cos\Theta_{c,s} = \frac{a_1(U+V_s)}{a_1^2 + V_s U}\left(1 + \frac{\Phi_{c,s}}{s_{c,s}}\right) \tag{21a}$$

and
$$\sin\Theta_{c,s} = \left(1 - \frac{a_1^2(U+V_s)^2}{(a_1^2+V_s U)^2}\right)^{\frac{1}{2}} \left(1 - \frac{1}{2}\frac{(U+V_s)^2}{a_1^2\beta^2\beta_s^2}\frac{\Phi_{c,s}}{s_{c,s}} + \dots\right), \tag{21b}$$

and $\Theta_{c,s}$ is approaching the critical value.

Substituting (21) into (14) gives

$$\mathscr{R}(\Theta_{c,s}) = -1 - \frac{2\hat{B}(1+V_s U/a_1^2)}{\hat{A}(1+V_s U/a_1^2) - \beta^2\beta_s^2}\frac{\Phi_{c,s}}{s_{c,s}} + \dots. \tag{22}$$

Thus, $1 + \mathscr{R}(\Theta_{c,s})$ decays as $s_{c,s}^{-1}$ and the leading terms in (13) and (16) decay as $s^{-\frac{3}{2}}$ or $t^{-\frac{3}{2}}$ for large times.

The spherical case is similar, with a wave emitted at the shock decaying as $s_{s,s}^{-2}$ or t^{-2}, a wave emitted far behind the shock and just reaching it decaying as s^{-1} or t^{-1} and a wave whose angle of incidence is approaching the critical value decaying as s^{-2} or t^{-2}.

5. Conclusions

The perturbations of a normal shock moving into a uniform medium are determined by the pressure perturbations in the flow behind the shock. These pressure perturbations are the result of both incident and reflected waves. The reflexion coefficient of a shock is asymptotic to minus one as the angle of incidence of the wave approaches the critical angle. Thus, if waves are generated behind the shock and subsequently catch up and interact with it, the shock perturbations will initially decay as the incident waves, as $t^{-\frac{1}{2}}$ or t^{-1}, for cylindrical and spherical waves respectively, before switching to a $t^{-\frac{3}{2}}$ or t^{-2} decay as the waves approach the critical angle of incidence, where the reflexion coefficient approaches minus one. Waves generated at the shock, as occur in inviscid shock diffraction and reflexion processes, will always propagate at the critical angle and will always be cancelled to lowest order by their reflexions. Thus they will always decay as $t^{-\frac{3}{2}}$ for cylindrical waves or t^{-2} for spherical waves. This is in agreement with previous analytical investigations of shock stability.

If the shock is perturbed by a continuous influx of waves from a source some distance behind it both types of behaviour occur together. As each wave front reaches the shock a period of $t^{-\frac{1}{2}}$ or t^{-1} decay occurs, in the two- or three-dimensional cases, while those waves which reached the shock earlier are decaying as $t^{-\frac{3}{2}}$ or t^{-2}. The slower decay, of course, will dominate and the resulting shock perturbations will behave in that manner. The experiments of Lapworth (1959) and Bowman (1966) have demonstrated this behaviour, for cylindrical waves, since waves were continuously being generated by the unsteady separation of the boundary layer on the obstacle, producing the predicted $t^{-\frac{1}{2}}$ decay.

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REFERENCES

Bowman, R. M. 1966 Investigation of shock front topography in shock tubes. Ph.D. thesis, California Institute of Technology, Pasadena.

Brillouin, J. 1955 Réflexion et réfraction d'ondes acoustiques par une onde de choc. *Acoustica*, **5**, 149. (Trans. 1957 Reflection and refraction of acoustic waves by a shock. *N.A.C.A. Tech. Memo.* no. 1409.)

Briscoe, M. G. & Kovitz, A. A. 1968 Experimental and theoretical study of the stability of plane shock waves reflected normally from a perturbed flat wall. *J. Fluid Mech.* **31**, 529.

D'iakov, S. P. 1958 The interaction of shock waves with small perturbations. *Sov. Phys., J. Exp. Theor. Phys.* **6**, 729.

Freeman, N. C. 1955 A theory of the stability of plane shock waves. *Proc. Roy. Soc.* A **228**, 341.

Freeman, N. C. 1957 On the stability of plane shock waves. *J. Fluid Mech.* **2**, 397.

Friedlander, F. G. 1946 Simple progressive solutions of the wave equation. *Proc. Camb. Phil. Soc.* **43**, 360.

Johnson, W. R. & Laporte, O. 1958 Interaction of cylindrical sound waves with a stationary shock wave. *Phys. Fluids*, **1**, 82.

KONTOROVICH, V. M. 1959 Reflection and refraction of sound by shock waves. *Sov. Phys., Acoustics*, **5**, 320.

KOVITZ, A. A. & BRISCOE, M. G. 1968 Stability of a plane shock wave in free space and in the presence of rigid or interfacial boundaries. *J. Acoust. Soc. Am.* **45**, 1157.

LAPWORTH, M. C. 1959 An experimental investigation of plane shock waves. *J. Fluid Mech.* **6**, 469.

LIGHTHILL, M. J. 1949 The diffraction of a blast. I. *Proc. Roy. Soc.* A **198**, 454.

MCKENZIE, J. F. & WESTPHAL, K. O. 1968 Interaction of linear waves with oblique shock waves. *Phys. Fluids*, **11**, 2350.

MOORE, F. K. 1954 Unsteady oblique interaction of a shock wave with a plane disturbance. *N.A.C.A. Rep.* no. 1165.

NIKOLAEV, I. M. 1965 Solution for a plane shock wave moving through a lightly curved interface of two media. *J. Appl. Math. Mech.* **29**, 785.

SEARS, W. R. 1954 Small perturbation theory. In *General Theory of High-Speed Aerodynamics*, p. 61. Princeton University Press.

VAN MOORHEM, W. K. 1971 The interaction of two dimensional acoustic waves with plane shocks. Ph.D. thesis, Cornell University, Ithaca, New York.

ZAIDEL, R. M. 1960 Shock wave from a slightly curved piston. *J. Appl. Math. Mech.* **24**, 316.